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Some Effects of Finite Particle Volume on the Dynamics of Gas-Particle Mixtures

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In the analysis of two-phase flows in which a gas carries along a large number of small particles, the volume of the particles is customarily assumed to be negligible. This assumption often is well justified, but if either the mass fraction of the particles or the gas density is sufficiently high, the particle volume fraction may become significant. It is thus important to establish the conditions at which this parameter should be included in a flow analysis. The particles may be considered as incompressible by comparison with the gas, so that a finite particle volume fraction appears as an additional variable in the basic equations. Consequences of a finite particle volume are examined for isentropic changes of the mixture, for frozen and equilibrium speed of sound, and for both the frozen flow immediately behind a shock wave and the equilibrium flow that is established further downstream. The errors that would result from neglecting the particle volume range from insignificant to large. For example, even for gas-particle density ratios as low as 10^{-3}, the equilibrium flow velocity behind a shock front is quite sensitive to changes of the volume of the particles if they represent more than one-half of the mass of the mixture.

Nomenclature

a = speed of sound of the gas phase
b = covolume of a Van der Waals gas
c = specific heat of particle material
c_p, c_v = specific heats of gas phase at constant pressure and constant volume
D = particle diameter
d = density of particle material
e_M = internal energy of mixture per unit mass
h_M = enthalpy of mixture per unit mass
$\mathcal{L} = 2.69 \times 10^{19}$ cm^{-3} = Loschmidt number
M_S = shock Mach number
$\mathfrak{M}_G$, $\mathfrak{M}_P$, $\mathfrak{M}_M$ = molecular weight of gas, particle, and mixture
m = mass flow rate of gas per unit area
n = mass flow rate of particles per unit area
p = pressure
p_G = partial pressure of gas phase
Q = heat
R = gas constant of gas phase
T = temperature of gas phase or of equilibrium mixture
u = velocity of gas phase or of equilibrium mixture
v = particle velocity if different from gas velocity
α = equilibrium speed of sound in the mixture
$\gamma = c_p/c_v$ = ratio of specific heats of gas phase
Γ = ratio of specific heats of mixture
$\delta = c/c_p$ = relative specific heat
ϵ = particle volume fraction
η = particle loading ratio
$\mu = 1.66 \times 10^{-24}$g = mass of hydrogen atom
ρ = density of gas phase
ρ_M = density of mixture
σ_G, σ_P = concentration of gas and particles
τ = particle temperature if different from gas temperature
φ = particle mass fraction

Subscripts

0 = standard values or conditions upstream of shock wave
f = frozen conditions behind shock wave
e = equilibrium conditions behind shock wave

Introduction

GAS flows that transport clouds of suspended solid particles or liquid droplets through a duct are of great technical importance. The mathematical analysis of such two-phase flows is considerably more difficult than that of pure gas flows, and one of the usual simplifying assumptions is that the volume occupied by the particles can be neglected. In many

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important cases, the particles represent less than one-half of the mass of the gas-particle mixture, and the density of the particle material is more than a thousand times larger than the gas density. Under such conditions, the particle volume fraction is of the order of only 10^{-4}, and the assumption of a negligible particle volume is then well satisfied. One important consequence of this assumption is that equilibrium flows of mixtures of particles with a perfect gas can be analyzed like flows of a perfect gas that has the density and specific heats of the mixture. Consequently, published analyses have been concerned mainly with various aspects of deviations from equilibrium that result from the particle velocity and temperature being unable to follow rapid changes in the gas phase. Examples are investigations of steady flow through nozzles (e.g., Refs. 1–5), flow behind shock waves (e.g., Refs. 2, 4, and 6–9), interactions between particles,[10] or propagation of arbitrary pressure waves.[11]

At high gas densities (high pressures) or at high particle mass fractions, the particle volume fraction may become sufficiently large so that it should not be neglected. It is therefore of interest to investigate how the results of flow computations are affected. One may then evaluate the conditions under which the particle volume may be neglected without introducing significant errors. Since the particles may be considered as incompressible in comparison with the gas, the particle volume fraction enters into the basic flow equations as an additional variable. Even equilibrium flows then cannot be treated as perfect-gas flows. In the following, a start is made toward investigating the consequences of finite particle volume. The discussion is limited to certain aspects of the thermodynamic properties of gas-particle mixtures and to frozen and equilibrium flow behind shock waves. Relaxation phenomena that result from velocity and temperature lags of the particles will not be considered.

Basic Properties of Gas-Particle Mixtures

Only a few assumptions are needed for the analytical work described in the following.

1) The gas is treated as a perfect gas with constant specific heats.

2) The density and specific heat of the particle material are constant.

3) No mass transfer takes place between the gas and the particles. (Phenomena associated with condensation, evaporation, or chemical reactions are excluded from the present discussion.)

4) Flows are treated as one-dimensional continuum flows. This assumption also implies that the size of the particles and their average separation are negligible as compared with significant dimensions of the flow field.

5) The random motion of the particles does not contribute to the pressure. This assumption implies that the number of particles is negligible compared with the number of gas molecules in the same volume. Since a finite particle volume often is the consequence of a large particle mass fraction, the limits of validity of this assumption are investigated separately.

Since only equilibrium and frozen conditions are considered in the following discussion, there is no need to make any assumptions about the drag and heat-transfer coefficients that characterize the interaction between the two phases in relaxation regions.

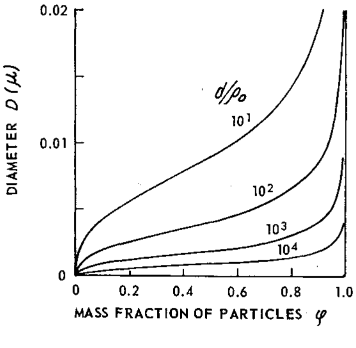

Fig. 1 Particle diameter below which the particle contribution to the pressure exceeds 1%.

In dealing with gas-particle mixtures with finite particle volume, one must distinguish between the gas density ρ (mass of gas in a unit volume of gas) and the gas concentration σ_G (mass of gas in a unit volume of the gas-particle mixture). Similarly, one must distinguish between the constant density of the particle material d and the variable particle concentration σ_P. The density of the mixture ρ_M, that is, the total mass in a unit volume of the mixture, must also be considered. As long as the particle volume can be neglected, the gas density equals the gas concentration, so that either definition may be used, and in fact, both may be found in the literature.

The mass fraction of the particles is given by

$$\varphi = \sigma_P/(\sigma_P + \sigma_G) \tag{1}$$

Some of the relationships may be expressed more conveniently if the loading ratio

$$\eta = \sigma_P/\sigma_G = \varphi/(1 - \varphi) \tag{2}$$

is used. Let the volume fraction of the particles be denoted by ϵ, so that

$$\sigma_P = \epsilon d \qquad \sigma_G = (1 - \epsilon)\rho \tag{3}$$

From Eqs. (2) and (3), it follows then that

$$\eta(\rho/d) = \epsilon/(1 - \epsilon) = \epsilon + \text{higher-order terms of } \epsilon \tag{4}$$

This equation shows that the particle volume fraction may become significant if the density ratio, or the loading ratio, or both, become sufficiently large. In many cases, ϵ still may be treated as a small quantity, and the linearized form of Eq. (4) may be used to simplify some of the equations. Note that, under equilibrium conditions, the mass fraction of the particles is constant whereas the volume fraction is variable.

It follows from the foregoing that the mixture density is related to the densities of the two phases by

$$\rho_M = \sigma_G + \sigma_P = (1 - \epsilon)\rho/(1 - \varphi) \tag{5}$$

but since ϵ is not a constant property of the mixture, it may be more convenient to write

$$1/\rho_M = [(1 - \varphi)/\rho] + (\varphi/d) \tag{6}$$

Equations (5) and (6) may be combined to

$$\epsilon = \varphi\rho_M/d \tag{7}$$

Before proceeding further, it is necessary to verify the assumption that the contribution of the particles to the pressure can be neglected. To obtain an estimate of this contribution, the particles are considered as large molecules of a second gas. It then is possible to determine the conditions for which the particle pressure of this "gas" reaches some selected level, say 1% of the pressure in the mixture.

The mass of a particle of diameter D is $\pi D^3 d/6$, so that the molecular weight of this "gas" becomes

$$\mathcal{M}_P = \pi D^3 d/6\mu \tag{8}$$

where $\mu = 1.66 \times 10^{-24}$ g is the mass of a hydrogen atom. Since the molecular weight of the mixture is given by

$$1/\mathcal{M}_M = (\varphi/\mathcal{M}_P) + [(1 - \varphi)/\mathcal{M}_G] \tag{9}$$

the ratio of the partial pressure of the gas p_G to the total pressure in the mixture becomes

$$\frac{p_G}{p} = \frac{(1 - \varphi)/\mathcal{M}_G}{(\varphi/\mathcal{M}_P) + (1 - \varphi)/\mathcal{M}_G} \tag{10}$$

If Eq. (8) is introduced into the last equation, this pressure ratio may be written in the form

$$\frac{p_G}{p} = \left(1 + \frac{\varphi}{1 - \varphi}\,\mathcal{M}_G\,\frac{6\mu}{\pi D^3 d}\right)^{-1} \tag{11}$$

Since the gas density at standard temperature and pressure is given by $\rho_0 = \mu\mathfrak{M}_G\mathcal{L}$, where $\mathcal{L} = 2.69 \times 10^{19}$ cm^{-3} is the Loschmidt number, the molecular weight of the gas may be eliminated from Eq. (11). Therefore, if the contribution of the particles to the pressure is to be less than 1% of the total pressure ($p_G/p \geq 0.99$), the particle diameter (measured in microns) must satisfy the condition

$$D \geq 0.0192[(d/\rho_0)(1 - \varphi)/\varphi]^{-1/3} \tag{12}$$

This relationship is shown in Fig. 1, and it can be seen that the particle contribution to the pressure can be neglected even at extremes of the density ratio and mass fraction as long as the particle diameter is larger than a few hundredths of a micron. This is not much larger than the size of condensation nuclei. The assumption that the random motion of the particles does not contribute to the pressure is therefore well justified.

The conditions of an equilibrium gas-particle mixture of specified materials are completely determined by four variables, such as the velocity u, temperature T, pressure p, and particle mass fraction φ. By the use of Eqs. (1–7) and the equation of state for a perfect gas

$$p = \rho RT \tag{13}$$

the state of the mixture may also be characterized by other sets of variables. The particular choice is dictated only by the convenience of the resultant relationships. If the mixture is not in equilibrium, as in frozen flow, the symbols u and T will refer to the gas only, whereas the particle velocity and temperature denoted by v and τ represent two additional variables.

Thermodynamics of Gas-Particle Mixtures

It is instructive to derive a few basic thermodynamic relationships because they clearly bring out the effects of a finite particle volume. If the gas density in Eq. (13) is eliminated with the help of Eqs. (6), or (6) and (7), one obtains the equation of state of a gas-particle mixture

$$p = \frac{\rho_M(1 - \varphi)RT}{1 - \varphi\rho_M/d} = \frac{\rho_M(1 - \varphi)RT}{1 - \epsilon} \tag{14}$$

The appearance of the variable particle volume fraction in the denominator shows that the deviations from the perfect-gas law are a direct consequence of a finite value of ϵ. For $\epsilon = 0$, the mixture behaves like a perfect gas with the modified gas constant $(1 - \varphi)R$, as was already noted in previous publications (e.g., Refs. 2, 4, 7, and 9).

The internal energy of the mixture is not affected by the particle volume and is given by the properly weighted average of that of the two constituents

$$e_M = (1 - \varphi)\, c_v\, T + \varphi c\tau \tag{15}$$

where c_v is the specific heat of the gas at constant volume and c that of the particles. For the sake of generality, the gas and the particles are assumed to have different temperatures. For the enthalpy one obtains then

$$\begin{aligned} h_M = e_M + (p/\rho_M) &= (1 - \varphi)[c_vT + (p/\rho)] + \\ &\qquad \varphi[c\tau + (p/d)] \\ &= (1 - \varphi)c_pT + \varphi[c\tau + (p/d)] \end{aligned} \tag{16}$$

by combining Eqs. (6) and (15). In this form, the enthalpy of the mixture represents the weighted average of the enthalpies of the two phases, but it is to be noted that the enthalpy contribution of the particles depends on the pressure of the gas phase. From Eqs. (2) and (4), one can easily show that the ratio of the two pressure terms in Eq. (16) is equal to $\epsilon/(1 - \epsilon)$. The pressure term in the enthalpy of the particles therefore appears only if the particle volume fraction cannot be neglected.

The specific heat of the equilibrium mixture at constant volume is given by $(\partial e_M/\partial T)_{\rho M}$, and that at constant pressure is given by $(\partial h_M/\partial T)_p$. Equations (15) and (17) therefore yield the ratio of the specific heats for the gas-particle mixture as

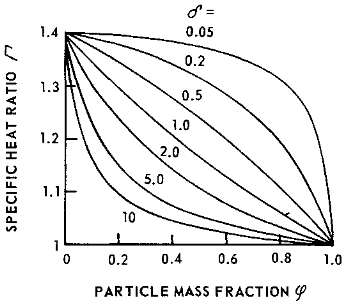

Fig. 2 Ratio of the specific heats of a gas-particle mixture for a diatomic gas.

$$\Gamma = \frac{(1 - \varphi)c_p + \varphi c}{(1 - \varphi)c_v + \varphi c} = \gamma\, \frac{1 + \eta\delta}{1 + \gamma\eta\delta} \tag{17}$$

where Eq. (2) and the relative specific heat $\delta = c/c_p$ have been used. This equation shows that Γ is always smaller than γ and approaches 1 for large loading ratios. It also indicates that Γ is independent of the particle volume fraction. The relationship between Γ, φ, and δ is represented in Fig. 2 for $\gamma = 1.4$ and several values of δ which essentially cover the entire range of interest.

Equations for isentropic changes of state of a gas-particle mixture may be derived from the first law of thermodynamics

$$dQ = de_M - p d\rho_M/\rho_M^2 \tag{18}$$

After setting $dQ = 0$ and substituting for p and e_M from Eqs. (14) and (15), one obtains

$$\frac{(1 - \varphi)c_v + \varphi c}{(1 - \varphi)R}\,\frac{dT}{T} = \frac{1}{1 - \varphi\rho_M/d}\,\frac{d\rho_M}{\rho_M}$$

Integration of this equation and use of Eq. (17) yields

$$T[\rho_M/(1 - \varphi\rho_M/d)]^{-(\Gamma-1)} = \text{const} \tag{19}$$

Since the denominator is equal to $1 - \epsilon$, according to Eq. (7), it is again found that the equation takes the same form as that for a pure gas, if the particle volume is negligible. The factor $1 - \epsilon$ introduces the correction required because the particles occupy a finite volume without participating in the volume change of the mixture. Because of Eq. (6), it is possible to rewrite Eq. (19) as

$$T\rho^{-(\Gamma-1)} = \text{const} \tag{20}$$

and if either ρ or T is eliminated from this equation with the help of Eq. (13), one obtains also

$$T^{-(\Gamma-1)/\Gamma} = \text{const} \tag{21}$$

and

$$p\rho^{-\Gamma} = \text{const} \tag{22}$$

Equations (20–22) are exact and do not require the assumption that ϵ be a small quantity. Thus, isentropic changes of state of a gas-particle mixture are most conveniently computed by using the ratio of the specific heats of the mixture and those variables that describe the state of the gas phase.

Another quantity of importance is the equilibrium speed of sound in a gas-particle mixture, denoted by α. It is defined as

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho_M}\right)_e = \left(\frac{\partial p}{\partial \rho}\right)_e \left(\frac{d\rho}{d\rho_M}\right)_e \tag{23}$$

where the subscript e indicates that the derivative must be evaluated for equilibrium (isentropic) changes of state. The first factor in this equation follows from Eqs. (22) and (13) as

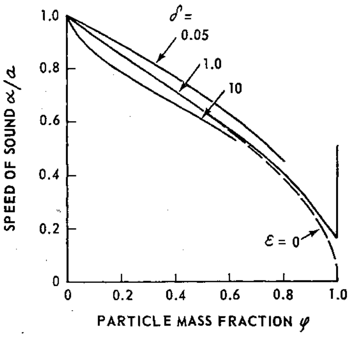

Fig. 3 Equilibrium speed of sound in a mixture of particles and a diatomic gas for $\rho/d = 0.01$.

ΓRT. The second factor, which may be evaluated by forming the logarithmic differentials of Eqs. (4) and (5) and then eliminating $d\epsilon$, is found to be $(1 - \varphi)/(1 - \epsilon)^2$. If these results are substituted into Eq. (23) and use is made of Eqs. (2, 4, and 17), the equilibrium speed of sound is obtained in terms of the speed of sound in the gas phase $a^2 = \gamma RT$ as

$$\left(\frac{\alpha}{a}\right)^2 = \frac{(1 + \eta\delta)(1 + \eta\rho/d)^2}{(1 + \eta)(1 + \gamma\eta\delta)} = \frac{\Gamma}{\gamma(1 + \eta)(1 - \epsilon)^2} \tag{24}$$

The first form of this equation indicates that α depends on the density (pressure) of the gas, and the second form clearly shows that the particle volume fraction increases the speed of sound according to

$$\alpha/\alpha_{\epsilon=0} = (1 - \epsilon)^{-1} \tag{25}$$

For negligible ϵ, Eq. (24) corresponds to relationships given in previous investigations (e.g., Refs. 2, 4, 6 and 9). Equations equivalent to Eq. (24) have also been derived for the speed of sound in liquids that contain gas bubbles (e.g., Refs. 12–14). As long as ϵ is sufficiently small, the speed of sound in the mixture is always smaller than that in the gas phase, but α increases to infinity (the speed of sound in the incompressible solid) as ϵ approaches 1. The existence of the resulting minimum is well known, but it is generally discussed for liquids containing air bubbles.[14] The limit $\epsilon = 1$, of course, cannot be reached in a mixture of a gas with solid particles since dense packing of spheres corresponds approximately to $\epsilon = 0.74$. For the representative case $\delta = 1$, it is not difficult to derive from Eq. (24) that the minimum value of the speed of sound is given by

$$\left(\frac{\alpha}{a}\right)^2_{\min} = \frac{4}{\gamma}\frac{\rho}{d}\left(1 - \frac{1}{\gamma}\frac{\rho}{d}\right) \simeq \frac{4}{\gamma}\frac{\rho}{d} \tag{26}$$

and is reached when

$$\eta = (d/\rho) - (2/\gamma) \simeq d/\rho \tag{27}$$

corresponding to

$$\epsilon = \tfrac{1}{2}\ [(\gamma d/\rho) - 2]/[(\gamma d/\rho) - 1] \simeq \tfrac{1}{2} \tag{28}$$

In most cases, the ratio d/ρ is quite large, and the indicated approximate expressions may be used.

The relationship between α/a and φ (rather than η) is plotted in Fig. 3 for $\gamma = 1.4$ and $\rho/d = 0.01$. For $\delta = 1$, the solid curve represents Eq. (24), and the broken line indicates the results obtained if the particle volume fraction is neglected. The speed of sound decreases markedly with increasing particle concentration, reaches a minimum at approximately $\varphi = 0.99$ ($\eta \simeq d/\rho = 100$), and then rapidly rises toward infinity. The curve for $\epsilon = 0$ goes to zero as the limit $\varphi = 1$ is approached and is independent of the density ratio ρ/d. It begins to deviate noticeably from the exact curve when the particle mass fraction exceeds 0.5. For values of ρ/d smaller than 0.01, this deviation would appear at larger mass fractions. Curves are also plotted for the maximum and minimum values of δ that ordinarily might be encountered. They exhibit a behavior similar to that for $\delta = 1$, but only part of the range of φ is covered in the figure for the sake of clarity.

It is interesting to note that the speed of sound in a Van der Waals gas without intermolecular forces exceeds that of the corresponding perfect gas by the factor $1 + b\rho$, where b is the covolume.[15] Since $b\rho$ thus represents the volume fraction taken up by the molecules, this result agrees with Eq. (25) to the first order in ϵ.

Shock Transition

A shock wave that propagates through a gas-particle mixture creates three distinct flow regions, namely, the frozen flow immediately behind the shock front, the equilibrium flow far behind the shock front, and a relaxation zone where the flow gradually changes from frozen to equilibrium conditions. Only the first two of these conditions will be discussed here.

In shock-fixed coordinates, the flow is steady, and the equilibrium conditions upstream of the shock wave are indicated by subscript 0. The conservation of mass, momentum, and energy may be expressed as

$$m = (1 - \epsilon)\rho\mu = (1 - \epsilon_0)\rho_0 u_0 \tag{29}$$

$$n = \epsilon dv = \epsilon_0 d u_0 \tag{30}$$

$$mu + nv + p = (m + n)u_0 + p_0 \tag{31}$$

$$m(\tfrac{1}{2}u^2 + c_pT) + n(\tfrac{1}{2}v^2 + c\tau + p/d) = \tfrac{1}{2}(m + n)u_0^2 + (mc_p + nc)T_0 + np_0/d \tag{32}$$

where m and n represents the mass flow rates of the gas and of the particles per unit area. Note that the mass flow ratio n/m is constant and, according to Eq. (4), equal to the loading ratio η whenever u and v are equal. For $\epsilon = 0$, the conservation equations reduce to their customarily used form.[2,4,6–9] The right-hand sides of the equations are known if the velocity u_0 and the mixture upstream of the shock wave are prescribed. The left-hand sides then contain the seven unknowns ρ, p, u, v, T, τ, and ϵ. In addition to the equation of state, two further conditions are therefore needed that depend on the region in which the flow is to be evaluated.

Frozen Flow

The transport processes of viscous drag and heat transfer lead to relaxation zones that are very large when compared with the size of the particles,[2, 4, 6–8] so that their contribution during the extremely short time of particle transition through the shock front can be neglected. Consequently, the customarily used condition for the frozen particle temperature still applies even if the particle volume is treated as finite, and

$$\tau_f = T_0 \tag{33}$$

While the particles are crossing the shock front, they are given an impulsive change of velocity that may be determined by integration of the equation of motion. The result is given by[16]

$$v_f^2 = u_0^2 - 2(\rho_0/d)\ (p_f - p_0)/\rho_0 \tag{34}$$

and indicates a velocity change that depends on the density ratio ρ_0/d, but not on the loading ratio. In the following, the particle volume fraction will be considered to be small enough so that powers of ϵ higher than the first can be neglected. Unusually extreme conditions of appreciable density ratio combined with very high loading ratio are thereby excluded. Equation (34) then may be linearized and yields, after combination with Eq. (30),

$$v_f = u_0 - (p_f - p_0)\epsilon_0/n \tag{35}$$

and

$$\epsilon_f = \epsilon_0 \tag{36}$$

to the first order of ϵ. Equations (33), (35), and (36) thus determine the frozen particle conditions. If they are substituted into Eqs. (29), (31), and (32), and use is made of the relationship $n/d = \epsilon_0 u_0$ from Eq. (30), one obtains the same conservation equations that apply for a pure gas. The effects of the reduced flow rate (by the factor $1 - \epsilon$) and those of the impulsive interaction between the two phases thus cancel each other to the first order of ϵ, and the frozen gas conditions are given by the Rankine-Hugoniot shock equations for the gas alone.

In the limiting case of an extremely weak shock wave, the velocity becomes equal to the frozen speed of sound which, in view of the foregoing, is equal to the speed of sound in the gas phase upstream of the shock wave.

Equilibrium Flow

The two conditions needed to complete the system of Eqs. (13) and (29–32) for equilibrium flow are simply

$$v_e = u_e \qquad \text{and} \qquad \tau_e = T_e \tag{37}$$

After systematically eliminating one variable after another, a result is obtained which is most concisely written in terms of the shock Mach number defined as $M_S = u_0/\alpha_0$ and the specific-heat ratio Γ of the mixture, where α and Γ are determined by Eqs. (17) and (24). The equilibrium velocity then is given by

$$\frac{u_e}{u_0} = \frac{(\Gamma - 1)M_S^2 + 2 + 2\epsilon_0(M_S^2 - 1)}{(\Gamma + 1)M_S^2} \tag{38}$$

and other variables are obtained readily as

$$\epsilon_e = \epsilon_0 u_0/u_e \tag{39}$$

$$\rho_e/\rho_0 = [(1 - \epsilon_0)/(1 - \epsilon_e)](u_0/u_e) \tag{40}$$

and

$$p_e/p_0 = 1 + (1 - \epsilon_0)\gamma(1 + \eta)(u_0 - u_e)u_0/a_0^2 \tag{41}$$

The variations of u_e/a_0 and of p_e/p_0 are plotted in Figs. 4 and 5 as functions of u_0/a_0 for $\gamma = 1.4$, $\delta = 1$, and for several values of η and ϵ_0. Shock transition is possible only if $M_S \geq 1$, and this condition can be satisfied even for $u_0 < a_0$ because $\alpha_0 < a_0$. For $u_0 < a_0$, transition takes the form of a dispersed shock wave (e.g., Refs. 4, 8, and 9) without a discontinuous shock front. Equations (38) and (41) indicate that $u_e = u_0$ and $p_e = p_0$ for $M_S = 1$, and this limit is also shown in Fig. 4. The minimum of u_e is located where u_0 satisfies the equation

$$\left(\frac{u_0}{a_0}\right)^2 = \frac{2(1 + \eta\delta)}{(1 + \eta)(1 - \epsilon_0)[\gamma - 1 + 2\epsilon_0(1 + \gamma\eta\delta)]} \tag{42}$$

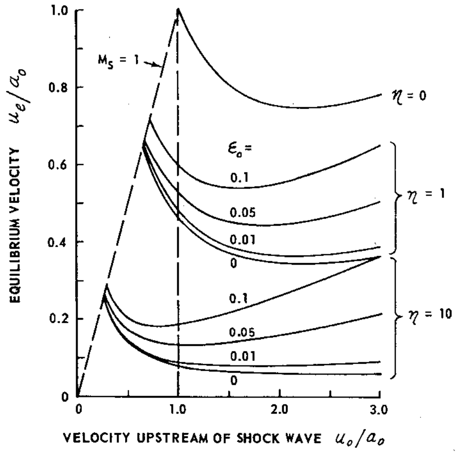

Fig. 4 Equilibrium velocity behind a shock wave for $\gamma = 1.4$ and $\delta = 1$.

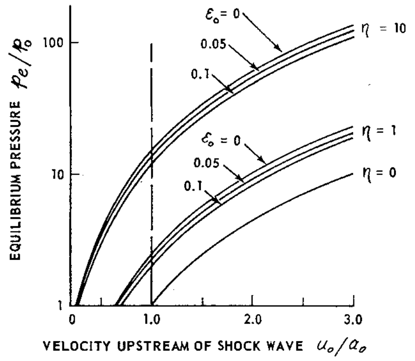

Fig. 5 Equilibrium pressure behind a shock wave for $\gamma = 1.4$ and $\delta = 1$.

which shows that the minimum is shifted toward lower values of u_0 if ϵ_0 is increased.

Discussion and Conclusions

The analysis of flows of gas-particle mixtures is considerably complicated if the particle volume cannot be neglected. It is shown that the random motion of the particles contributes less than 1% to the pressure as long as the particles are larger than a few hundredths of a micron. Such particles are not much larger than condensation nuclei, and the customary assumption that the particles do not contribute to the pressure is therefore valid for practically all of the conditions of importance. Since the particles may be treated as incompressible in comparison with the gas, their volume does not participate in volume changes of the mixture, and the particle volume fraction enters into the calculations as an additional variable. One important consequence is that mixtures of particles with a perfect gas do not follow the perfect-gas law.

The specific-heat ratio of the mixture always decreases with increasing mass fraction of the particles in a manner that depends significantly upon the relative specific heat of the two phases but is independent of the particle volume fraction. The equilibrium speed of sound also decreases with increasing particle mass fraction and is somewhat affected by the value of δ; it reaches a minimum at extremely high mass fractions, when the particles occupy about one-half of the mixture volume. For still higher mass fractions, the speed of sound would approach infinity rapidly. (If the particles had been assumed to be compressible, a more precise analysis would have yielded the speed of sound in the particle material as the upper limit.) If the particle volume in a mixture is neglected, the computed speed of sound is too small by the factor $1 - \epsilon$.

Different variables may be used to describe the state of the mixture, and it is shown that a judicious choice may lead to considerable simplification of some of the relationships. For example, the equations for isentropic compression of a mixture assume their most convenient form if one of the variables is the gas density, rather than the mixture density of the gas concentration.

Both frozen and equilibrium flows behind a shock wave have been analyzed. Frozen flow is characterized by the absence of transport processes (viscous drag and heat transfer) and not by the total absence of interaction between the two phases, since the impulse received by the particles while they are crossing the shock front must be taken into account. It is found that the frozen particle velocity depends directly upon the gas-particle density ratio and not on the particle volume fraction. To the first order of the particle volume fraction, the frozen gas flow is the same as that behind a shock wave in the pure gas, since the effect of the reduced flow rate then just cancels that of the impulse given to the particles. The corresponding frozen speed of sound is thus also unaffected by the presence of the particles.

Table 1 Dimensionless equilibrium velocity u_e/a_0 behind a shock wave for an upstream velocity $u_0 = 1.5\ a_0$ ($\gamma = 1.4$ and $\delta = 1$)

	Particle volume fraction			
Loading ratio	0	0.01	0.05	0.10
0	0.80	...	...	...
1	0.37	0.39	0.45	0.54
10	0.06	0.08	0.14	0.22

The effects of the particle volume on the equilibrium flow are demonstrated in Figs. 4 and 5, and a few of these results are collected in Table 1 to show the effects even more clearly. The large decrease of the flow velocity with increasing loading ratio is a direct consequence of the lowering of the speed of sound which corrresponds to an increase of the shock Mach number. The increase of the velocity with particle volume fraction is caused partly by the corresponding decrease of the shock Mach number and partly by a direct effect of the particle volume, as indicated by Eq. (38). It can be seen that even a small increase of the particle volume fraction leads to a marked increase of the velocity if the loading ratio is greater than one. If a volume fraction of 0.01 at a loading ratio of 10 were neglected, the computed velocity would be too low by about 25%. According to Eq. (4), this condition would require a gas-particle density ratio of only 10^{-3}.

The equilibrium pressure increases with the loading ratio and decreases with increasing particle volume. These changes can be explained by the same reasoning as for the gas velocity. As indicated in Fig. 5, the pressure decrease caused by the finite particle volume is almost independent of loading ratio and shock velocity; it amounts to about 20% for $\epsilon_0 = 0.1$.

It is evident from the presented results that the errors introduced in a flow analysis by neglecting the finite particle volume range from insignificant to large. It is not necessary to have extreme conditions in order to have the particle volume produce significant effects. For average gas and particle densities, such conditions may be reached when a loading ratio of 1 is exceeded. The influence of the particle volume on flows that involve velocity and temperature relaxation has not been considered in the present study, but, in view of the foregoing, it would be desirable to extend future investigations to such flows.

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